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The Herglotz–Zagier function, double zeta functions, and values of L-series

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Abstract

In this paper, we define L-series generalizing the Herglotz–Zagier function (Ber. Verhandl. Sächsischen Akad. Wiss. Leipzig 75 (3–14) (1923) 31, Math. Ann. 213 (1975) 153), and the double zeta function (First European Congress of Mathematics, Vol. II, Paris, 1992, pp. 497–512; Progress in Mathematics, Vol. 120, Birkhäuser, Basel, 1994) and evaluate them after meromorphic continuation at integer points in their extended domains. This is accomplished in three steps. First, when $\alpha : \mathbb{Z} \to \mathbb{C}$ is a periodic function and $h(n) = \sum_{j=1}^{n} j^{-1}$ are the harmonic numbers, we establish identities relating these series to the L-series

$$H(\alpha, s) = \sum_{n=1}^{\infty} \alpha(n)h(n)n^{-s}, \quad \mathrm{Re}(s) > 1,$$

and the Dirichlet L-function. Second, we prove that $H(\alpha, s)$ has a meromorphic continuation to $\mathbb{C}$ and evaluate $H(\alpha, s)$ at $s = -2l$ for each integer $l \geqslant 0$ in terms of Hurwitz zeta functions, generalized Euler's constants, a finite sum, and zeta functions closely resembling the Hurwitz zeta function. Third, we combine steps one and two with well-known facts concerning the Dirichlet L-function to obtain the desired evaluations. Our results for $H(\alpha, s)$ generalize previous work of Apostol and Vu (J. Number Theory 19 (1) (1984) 85) in the case α is identically one with period one.

Keywords: Herglotz–Zagier function; Double zeta function; Harmonic numbers; Meromorphic continuation; Values of L-series

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1. Introduction and statement of results

1.1. In [4], Apostol and Vu studied analytic and arithmetic properties of the zeta function

$$H(s,z)=\sum_{n=1}^{\infty} n^{-s}\sum_{j=1}^{n} j^{-z}, \quad \mathrm{Re}(s+z)>1, \quad \mathrm{Re}(s)>1,$$

an important special case of which is the L-series

$$H(s)=\sum_{n=1}^{\infty} h(n)n^{-s}, \quad \mathrm{Re}(s)>1,$$

whose coefficients $h(n)=\sum_{j=1}^{n} j^{-1}$ are given by the harmonic numbers. Apostol and Vu proved that $H(s)$ has a meromorphic continuation to $\mathbb{C}$ with a double pole at $s=1$ and simple poles at $s=0$ and $1-2r$ for each integer $r\geqslant 1$, and evaluated $H(s)$ at the negative even integers in terms of rational multiples of Bernoulli numbers.

When $\alpha:\mathbb{Z}\to\mathbb{C}$ is periodic with period N, we define the twist of $H(s)$ by

$$H(\alpha,s)=\sum_{n=1}^{\infty} \alpha(n)h(n)n^{-s}, \quad \mathrm{Re}(s)>1.$$

The series $H(\alpha,s)$ arose during the author's investigation of multiple L-series [19], and we subsequently found identities relating $H(\alpha,s)$ to L-series generalizing the Herglotz–Zagier function [13,23], and the double zeta function [24]. We sought to meromorphically continue these series and evaluate them at integer points in their extended domains, and to do this we needed results for $H(\alpha,s)$ analogous to those of Apostol and Vu for $H(s)$.

In Section 2, we prove that $H(\alpha,s)$ has a meromorphic continuation to $\mathbb{C}$ with a double pole at $s=1$ and possible simple poles at $s=0$ and $1-2r$ for each integer $r\geqslant 1$. Let $\alpha^0:\mathbb{Z}\to\mathbb{C}$ be identically 1 with period 1. In Section 3, we establish the following evaluation formula for $H(\alpha,s)$.

Theorem 1. *Suppose that $\alpha\neq\alpha^0$. For each integer $l\geqslant 0$,*

$$H(\alpha,-2l)=\sum_{d=1}^{6} g_d(\alpha,l,N), \tag{1.1}$$

where the terms g_1 and g_2 are complex linear combinations of values of the Hurwitz zeta function

$$\zeta(s,x)=\sum_{n=0}^{\infty} (s+x)^{-s}, \quad \mathrm{Re}(s)>1,$$

the terms g_3 and g_4 are complex linear combinations of values of the generalized Euler's constant

$$\gamma(x) = \lim_{n\to\infty}\left[\sum_{j=0}^{n}\frac{1}{j+x} - \log(n+x)\right],$$

the term g_5 is a finite sum, and the term g_6 is a complex linear combination of values of the zeta function

$$Z(s,r,r') = \sum_{q=0}^{\infty}(q+r)^{-1}(q+r')^{-s}, \quad \mathrm{Re}(s)>0,$$

where $(r,r')\in\mathbb{Q}_+\times\mathbb{Q}_+$.

The terms $\{g_d\}$ in Theorem 1 are defined in Section 3.

For $s\in\mathbb{C}$ and $n\in\mathbb{Z}_+$, define the binomial coefficient

$$\binom{s}{n} = \frac{s(s-1)\cdots(s-n+1)}{n!}$$

and let B_l be the lth Bernoulli number (see [3, p. 265]). In Section 3, we establish the following evaluation formula for $H(s)=H(\alpha^0,s)$.

Corollary 2. *For each integer $l\geqslant 1$,*

$$H(\alpha^0,-2l) = -\left[1+\binom{-2}{2l-1}\right]\frac{B_{2l}}{4l}.$$

Remark 3. The term $\binom{-2}{2l-1}$ is missing in the evaluation formula for $H(s)$ given in [4].

1.2. The Herglotz–Zagier function is defined by

$$F(x) = \sum_{n=1}^{\infty}\left[\frac{\Gamma'(nx)}{\Gamma(nx)} - \log(nx)\right]n^{-1}, \quad x>0,$$

where $\Gamma(s)$ is Euler's gamma function. Herglotz [13] and Zagier [23] used $F(x)$ to compute the constant term in the Laurent expansion at $s=1$ of the Dedekind zeta function of a narrow ideal class of a real quadratic field, thereby obtaining a Kronecker limit formula. Zagier's approach to this problem can be described as follows. Let K be a real quadratic field of discriminant D, B be a narrow ideal class of K, and $\zeta(B^{-1},s)$ be the Dedekind zeta function of the ideal class B^{-1}. First, Zagier obtained a decomposition

$$D^{s/2}\zeta(B^{-1},s)=\sum_{k=1}^{n} Z_{Q_k}(s),$$

where

$$Z_Q(s)=\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} Q(p,q)^{-s},\quad \mathrm{Re}(s)>1,$$

is the zeta function attached to the quadratic form $Q(p,q)=ap^2+bpq+cq^2$ with $a,b,c>0$ and $b^2-4ac=1$. Second, he used $F(x)$ to compute the constant term in the Laurent expansion of $Z_Q(s)$ at $s=1$, after meromorphic continuation, which when combined with the above decomposition led to his Kronecker limit formula.

Zagier also gave a detailed analysis of $F(x)$, explaining the contribution of $F(x)$ to his Kronecker limit formula, proving several functional equations for $F(x)$, and evaluating $F(x)$ and $F'(x)$ at $x=1$ (see [23, Section 7]). Recently, we evaluated $F(x)$ at $x=d$ for each integer $d\geqslant 1$ using an Euler-Maclaurin expansion for lattices above dimension one (see [20]). Ishibashi [15] has defined higher Herglotz–Zagier functions and used them to evaluate all of the Laurent coefficients of $Z_Q(s)$ at $s=1$ in closed form. His approach is related to earlier work of Deninger [7] on an analogue of the Chowla–Selberg formula for real quadratic fields.

Motivated by the many interesting properties of $F(x)$, we defined the following L-series generalizing $F(x)$,

$$F(\alpha,s,x)=\sum_{n=1}^{\infty}\alpha(n)\left[\frac{\Gamma'(nx)}{\Gamma(nx)}-\log(nx)\right]n^{-s},\quad \mathrm{Re}(s)>0,\quad x>0.$$

In Section 4, we prove that $F(\alpha,s,x)$ has a meromorphic continuation to $\mathbb{C}$ with possible simple poles at $s=0$ and $1-2r$ for each integer $r\geqslant 1$.

In order to relate $F(\alpha,s,x)$ to $H(\alpha,s)$, we define the L-series

$$L(\alpha,s)=\sum_{n=1}^{\infty}\alpha(n)n^{-s},\quad \mathrm{Re}(s)>1,$$

which generalizes the Dirichlet L-function and possesses many analogous properties. See for example work of Berndt and Schoenfeld [5], and [14, Chapter 2]. Let γ be Euler's constant and consider the identity (see [18, p. 6])

$$\frac{\Gamma'(n)}{\Gamma(n)}-\log(n)=h(n)-\log(n)-n^{-1}-\gamma,\quad n\in\mathbb{Z}_+. \tag{1.2}$$

Using Eq. (1.2), we obtain the formula

$$F(\alpha,s,1)=H(\alpha,s)+L'(\alpha,s)-L(\alpha,s+1)-\gamma L(\alpha,s). \tag{1.3}$$

Let χ be a non-principal, primitive Dirichlet character of conductor N and χ^0 be the principal Dirichlet character of conductor 1. Further, let $B_{\chi,l}$ be the lth generalized Bernoulli number and $\tau(\chi)$ be the Gauss sum associated to the character χ (these numbers are defined in Section 5 below). In Section 5, we use Eq. (1.3) to establish the following evaluation formulas for $F(\chi, s, 1)$.

Theorem 4. (A) *Suppose* $\chi \neq \chi^0$. *Then*

$$F(\chi, 0, 1) = \begin{cases} H(\chi, 0) + \frac{\tau(\chi)}{2} L(\bar{\chi}, 1) - L(\chi, 1) & \text{if } \chi(-1) = 1, \\ H(\chi, 0) + L'(\chi, 0) - L(\chi, 1) + \gamma B_{\chi,1} & \text{if } \chi(-1) = -1. \end{cases}$$

For each integer $l \geqslant 1$,

$$F(\chi, -2l, 1) = \begin{cases} H(\chi, -2l) + \dfrac{\tau(\chi)}{2} \dfrac{(2l)!}{(2\pi i)^{2l}} N^{2l} L(\bar{\chi}, 2l+1) + B_{\chi,2l}/2l & \text{if } \chi(-1) = 1, \\ H(\chi, -2l) + L'(\chi, -2l) + \gamma B_{\chi,2l+1}/(2l+1) & \text{if } \chi(-1) = -1. \end{cases}$$

(B) *For each integer* $l \geqslant 1$,

$$F(\chi^0, -2l, 1) = \frac{1}{2} \frac{(2l)!}{(2\pi i)^{2l}} \zeta(2l+1) + \left[1 - \binom{-2}{2l-1}\right] \frac{B_{2l}}{4l}.$$

We also defined the related L-series

$$G(\alpha, s, x) = \sum_{n=1}^{\infty} \alpha(n) \frac{\Gamma'(nx)}{\Gamma(nx)} n^{-s}, \quad \mathrm{Re}(s) > 1, \ \ x > 0.$$

In Section 4, we prove that $G(\alpha, s, x)$ has a meromorphic continuation to $\mathbb{C}$ with a possible double pole at $s = 1$ and possible simple poles at $s = 0$ and $1 - 2r$ for each integer $r \geqslant 1$.

By another application of Eq. (1.2) we obtain the formula

$$G(\alpha, s, 1) = H(\alpha, s) - L(\alpha, s+1) - \gamma L(\alpha, s). \tag{1.4}$$

In Section 5, we use Eq. (1.4) to establish the following evaluation formulas for $G(\chi, s, 1)$.

Theorem 5. (A) *Suppose that* $\chi \neq \chi^0$. *Then*

$$G(\chi, 0, 1) = \begin{cases} H(\chi, 0) - L(\chi, 1) & \text{if } \chi(-1) = 1, \\ H(\chi, 0) - L(\chi, 1) + \gamma B_{\chi,1} & \text{if } \chi(-1) = -1. \end{cases}$$

For each integer $l \geqslant 1$,

$$G(\chi, -2l, 1) = \begin{cases} H(\chi, -2l) + B_{\chi,2l}/2l & \text{if } \chi(-1) = 1, \\ H(\chi, -2l) + \gamma B_{\chi,2l+1}/(2l+1) & \text{if } \chi(-1) = -1. \end{cases}$$

(B) *For each integer* $l \geqslant 1$,

$$G(\chi^0, -2l, 1) = \left[1 - \binom{-2}{2l-1}\right] \frac{B_{2l}}{4l}.$$

1.3. The double zeta function is defined by (see [24])

$$\zeta_2(z,s) = \sum_{0<m<n} m^{-z} n^{-s}, \quad \mathrm{Re}(z+s) > 2, \quad \mathrm{Re}(s) > 1,$$

and the double zeta values are the complex numbers $\zeta_2(a,b)$ where a and b are positive integers with $b > 1$. The double zeta values have been studied extensively in recent years, arising in connection with arithmetic algebraic geometry [8,10–12,22], number theory [24,25], hyperbolic geometry [9], and quantum physics [6]. Despite the fact that $\zeta_2(z,s)$ has a meromorphic continuation to $\mathbb{C}^2$ (see [26]), relatively little is known about arithmetic properties of $\zeta_2(z,s)$ at points in its extended domain (see [1,2,19]).

In order to relate $\zeta_2(z,s)$ to $H(\alpha,s)$, we define the following double L-series generalizing $\zeta_2(z,s)$,

$$L_2(\alpha, z, s) = \sum_{0<m<n} \alpha(n) m^{-z} n^{-s}, \quad \mathrm{Re}(z+s) > 2, \quad \mathrm{Re}(s) > 1.$$

The double L-series can be expressed as

$$L_2(\alpha, z, s) = \sum_{n=1}^{\infty} \alpha(n) n^{-s} \sum_{m=1}^{n-1} m^{-z},$$

from which we obtain the formula

$$L_2(\alpha, 1, s) = H(\alpha, s) - L(\alpha, s+1). \tag{1.5}$$

In Section 5, we use Eq. (1.5) to establish the following evaluation formulas for $L_2(\chi, 1, s)$.

Theorem 6. (A) *Suppose* $\chi \neq \chi^0$. *Then*

$$L_2(\chi, 1, 0) = H(\chi, 0) - L(\chi, 1).$$

For each integer $l \geqslant 1$,

$$L_2(\chi, 1, -2l) = \begin{cases} H(\chi, -2l) + B_{\chi,2l}/2l & \text{if } \chi(-1) = 1, \\ H(\chi, -2l) & \text{if } \chi(-1) = -1. \end{cases}$$

(B) *For each integer* $l \geqslant 1$,

$$\zeta_2(1, -2l) = \left[1 - \binom{-2}{2l-1}\right] \frac{B_{2l}}{4l}.$$

From part (B) of Theorems 5 and 6, we see that $G(\chi^0, -2l, 1) = \zeta_2(1, -2l)$ for each integer $l \geqslant 1$.

2. Meromorphic continuation of $H(\boldsymbol{\alpha}, s)$

In this section we give two proofs that $H(\alpha, s)$ has a meromorphic continuation to $\mathbb{C}$. The first proof allows us to determine the possible poles and residues of $H(\alpha, s)$. The second proof involves an alternate expression for $H(\alpha, s)$ which is critical to the proof of Theorem 1.

We begin with the following lemma, both parts of which are well-known applications of the Euler–Maclaurin summation formula (see [17, p. 531]).

Lemma 7. *For each integer* $k \geqslant 1$, *let* $P_k(x)$ *be the periodic function with period* 1 *defined by the Fourier series*

$$P_k(x) = -\frac{1}{(2\pi i)^k} \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{e^{2\pi i l x}}{l^k}, \quad x \in \mathbb{R}.$$

Then the following identities are valid for all integers $n, k \geqslant 1$:

$$\sum_{j=1}^{n} j^{-1} = \log(n) + \gamma + \frac{1}{2n} - \sum_{r=1}^{k} \frac{B_{2r}}{2r} n^{-2r} + (2k+1)! \int_n^\infty \frac{P_{2k+1}(x)}{x^{2k+2}}\, dx, \tag{2.1}$$

$$\begin{aligned}\sum_{j=n}^{\infty} j^{-s} = &\frac{n^{-(s-1)}}{s-1} + \frac{n^{-s}}{2} + \sum_{r=1}^{k} \binom{s+2r-2}{2r-1} \frac{B_{2r}}{2r n^{s+2r-1}} \\ &- (2k+1)! \binom{s+2k}{2k+1} \int_n^\infty \frac{P_{2k+1}(x)}{x^{s+2k+1}}\, dx.\end{aligned} \tag{2.2}$$

Recall that the L-series $L(\alpha, s)$ has a meromorphic continuation to $\mathbb{C}$ with a possible simple pole at $s = 1$, and the Hurwitz zeta function $\zeta(s, x)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1 (for these facts see [14, pp. 41–42]).

Theorem 8. *The L-series* $H(\alpha, s)$ *has a meromorphic continuation to* $\mathbb{C}$ *with a double pole at* $s = 1$ *and possible simple poles at* $s = 0$ *and* $s = 1 - 2r$ *for each integer* $r \geqslant 1$.

Proof 1. Using the periodicity of α and an application of Lemma 7 (2.1), we obtain the following expression for $H(\alpha, s)$, valid for $\mathrm{Re}(s) > 1$ and each integer $k \geqslant 1$,

$$H(\alpha, s) = \sum_{m=1}^{N} \alpha(m) \sum_{l=0}^{\infty} \log\,(Nl+m)(Nl+m)^{-s} + \gamma L(\alpha, s) + \frac{1}{2} L(\alpha, s+1) - \sum_{r=1}^{k} \frac{B_{2r}}{2r} L(\alpha, s+2r) + R(\alpha, k, s). \tag{2.3}$$

Here the remainder term is given by the series

$$R(\alpha, k, s) = \sum_{n=1}^{\infty} \alpha(n) \left[(2k+1)! \int_{n}^{\infty} \frac{P_{2k+1}(x)}{x^{2k+2}} dx \right] n^{-s}.$$

Using multiplicativity of $\log(x)$, we obtain

$$\sum_{m=1}^{N} \alpha(m) \sum_{l=0}^{\infty} \log(Nl+m)(Nl+m)^{-s} = \log(N) L(\alpha, s) - N^{-s} \sum_{m=1}^{N} \alpha(m) \zeta'\left(s, \frac{m}{N}\right). \tag{2.4}$$

Then substituting Eq. (2.4) into Eq. (2.3) yields

$$H(\alpha, s) = (\log(N) + \gamma) L(\alpha, s) - N^{-s} \sum_{m=1}^{N} \alpha(m) \zeta'\left(s, \frac{m}{N}\right) + \frac{1}{2} L(\alpha, s+1) - \sum_{r=1}^{k} \frac{B_{2r}}{2r} L(\alpha, s+2r) + R(\alpha, k, s). \tag{2.5}$$

Since $P_{2k+1}(x)$ is bounded on $\mathbb{R}$, a straightforward estimate shows that $R(\alpha, k, s)$ is absolutely convergent for $\mathrm{Re}(s) > -2k$. As this holds for each integer $k \geqslant 1$, we conclude from Eq. (2.5) that $H(\alpha, s)$ has a meromorphic continuation to $\mathbb{C}$ with a double pole at $s = 1$ and possible simple poles at $s = 0$ and $1 - 2r$ for $r = 1, 2, \ldots$. $\square$

Remark 9. We conclude from Eq. (2.5) that $H(\alpha^0, s)$ has a double pole at $s = 1$ with residue γ, and simple poles at $s = 0$ and $s = 1 - 2r$ for $r = 1, 2, \ldots$, with residues $1/2$ and $-B_{2r}/2r$ for $r = 1, 2, \ldots$, respectively.

The strategy of the second proof is as follows. We use a modification of Lemma 7 (2.2) to express $H(\alpha, s)$ as the sum of four functions, and then prove that each of these functions has a meromorphic continuation to $\mathbb{C}$. For the first three functions we derive a recurrence relation which is used to express these functions in terms of Hurwitz zeta functions and series which are absolutely convergent in arbitrarily large half-planes, and for the fourth function we use properties of the remainder term in the modified Lemma 7 (2.2).

Proof 2. Let $\{a_n\}$ and $\{b_k\}$ be sequences of complex numbers. Then the following identities hold whenever the series is absolutely convergent,

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} a_n b_k = \sum_{n=1}^{\infty} b_n \sum_{k=n}^{\infty} a_k. \tag{2.6}$$

If $a_n = \alpha(n)n^{-s}$ and $b_k = k^{-1}$ in Eq. (2.6), we can use Eq. (2.6) and the periodicity of α to obtain the following expression for $H(\alpha, s)$, valid for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} H(\alpha, s) &= \sum_{n=1}^{\infty} \alpha(n)n^{-s} \sum_{k=1}^{n} k^{-1} \\ &= \sum_{n=1}^{\infty} n^{-1} \sum_{k=n}^{\infty} \alpha(k)k^{-s} \\ &= \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \alpha(n+m)n^{-1} \sum_{l=0}^{\infty} (Nl + n + m)^{-s} \\ &= N^{-s} \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \alpha(n+m)n^{-1} \sum_{l=0}^{\infty} \left(l + \frac{n+m}{N}\right)^{-s}. \end{aligned} \tag{2.7}$$

Next, by a slight modification of Lemma 7 (2.2), we obtain the following formula, valid for $\mathrm{Re}(s) > 1$ and each integer $k \geqslant 1$,

$$\begin{aligned} \sum_{l=0}^{\infty} \left(l + \frac{n+m}{N}\right)^{-s} = {} & \frac{((n+m)/N)^{-(s-1)}}{s-1} + \frac{((n+m)/N)^{-s}}{2} \\ & + \sum_{r=1}^{k} \binom{s+2r-2}{2r-1} \frac{B_{2r}}{2r((n+m)/N)^{s+2r-1}} \\ & - (2k+1)! \binom{s+2k}{2k+1} \int_0^{\infty} \frac{P_{2k+1}(x)}{(x + (n+m)/N)^{s+2k+1}}\, dx. \end{aligned} \tag{2.8}$$

Then substituting Eq. (2.8) into Eq. (2.7) yields the following expression for $H(\alpha, s)$, valid for $\mathrm{Re}(s) > 1$ and each integer $k \geqslant 1$,

$$\begin{aligned} H(\alpha, s) = {} & \frac{N^{-s}}{s-1} \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \alpha(n+m)n^{-1} \left(\frac{n+m}{N}\right)^{-(s-1)} \\ & + \frac{N^{-s}}{2} \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \alpha(n+m)n^{-1} \left(\frac{n+m}{N}\right)^{-s} \end{aligned}$$

$$
\begin{aligned}
&+ N^{-s} \sum_{r=1}^{k} \binom{s+2r-2}{2r-1} \frac{B_{2r}}{2r} \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \frac{\alpha(n+m)n^{-1}}{((n+m)/N)^{s+2r-1}} \\
&- (2k+1)! \binom{s+2k}{2k+1} \\
&\quad \times \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \alpha(n+m)n^{-1} \int_0^\infty \frac{P_{2k+1}(x)}{(x+(n+m)/N)^{s+2k+1}}\, dx \\
&= H_1(\alpha,s) + H_2(\alpha,s) + H_3(\alpha,s) + H_4(\alpha,s).
\end{aligned}
$$

Using the periodicity of α we obtain the following expression for H_1, valid for $\mathrm{Re}(s) > 1$,

$$
\begin{aligned}
H_1(\alpha,s) &= \frac{N^{-1}}{s-1} \sum_{m=0}^{N-1} \sum_{n=1}^{\infty} \alpha(n+m)n^{-1}(n+m)^{-(s-1)} \\
&= \frac{N^{-1}}{s-1} \sum_{m=0}^{N-1} \sum_{p=1}^{N} \alpha(m+p) \sum_{q=0}^{\infty} (Nq+p)^{-1}(Nq+m+p)^{-(s-1)} \\
&= \frac{N^{-(s+1)}}{s-1} \sum_{m=0}^{N-1} \sum_{p=1}^{N} \alpha(m+p) \sum_{q=0}^{\infty} \left(q+\frac{p}{N}\right)^{-1} \left(q+\frac{m+p}{N}\right)^{-(s-1)}. \qquad (2.9)
\end{aligned}
$$

Further, using the identity

$$
\frac{1}{q+\frac{p}{N}} = \frac{1}{q+\frac{m+p}{N}} + \frac{\frac{m}{N}}{\left(q+\frac{p}{N}\right)\left(q+\frac{m+p}{N}\right)},
$$

we obtain the following recurrence relation, valid for $\mathrm{Re}(s) > 1$ and each integer $a \geqslant 0$,

$$
\begin{aligned}
\sum_{q=0}^{\infty} \left(q+\frac{p}{N}\right)^{-1} \left(q+\frac{m+p}{N}\right)^{-(s-1)} &= \sum_{j=0}^{a} \left(\frac{m}{N}\right)^{j} \sum_{q=0}^{\infty} \left(q+\frac{m+p}{N}\right)^{-(s+j)} \\
&+ \left(\frac{m}{N}\right)^{a+1} \sum_{q=0}^{\infty} \left(q+\frac{p}{N}\right)^{-1} \left(q+\frac{m+p}{N}\right)^{-(s+a)}. \qquad (2.10)
\end{aligned}
$$

The first series on the right-hand side of Eq. (2.10) can be expressed in terms of Hurwitz zeta functions as follows. Recalling that $m = 0, \dots, N-1$ and $p = 1, 2, \dots N$, for $\mathrm{Re}(s) > 1 - j$ we have

$$
\sum_{q=0}^{\infty} \left(q+\frac{m+p}{N}\right)^{-(s+j)} = \zeta\left(s+j, \frac{m+p}{N}\right) \qquad (2.11)
$$

if $0 < m+p \leqslant N$, and

$$
\sum_{q=0}^{\infty} \left(q+\frac{m+p}{N}\right)^{-(s+j)} = \zeta\left(s+j, \frac{m+p}{N} - 1\right) - \left(\frac{m+p}{N} - 1\right)^{-(s+j)} \qquad (2.12)
$$

if $N < m+p \leqslant 2N-1$.

Define the set $L = \{0, \ldots, N-1\} \times \{1, \ldots, N\}$ and the function

$$A(\alpha, s, a) = \sum_{\substack{(m,p)\in L \\ 0 < m+p \leqslant N}} \alpha(m+p) \sum_{j=0}^{a} \left(\frac{m}{N}\right)^{j} \zeta\left(s+j, \frac{m+p}{N}\right) + \sum_{\substack{(m,p)\in L \\ N < m+p \leqslant 2N-1}} \alpha(m+p) \sum_{j=0}^{a} \left(\frac{m}{N}\right)^{j} \zeta\left(s+j, \frac{m+p}{N} - 1\right).$$

Then substituting Eq. (2.10) into Eq. (2.9) and applying Eqs. (2.11) and (2.12) to the result yields the following expression for H_1, which is meromorphic in the half-plane $\mathrm{Re}(s) > -a$,

$$H_1(\alpha, s) = \frac{N^{-(s+1)}}{s-1}\left[A(\alpha, s, a) - \sum_{\substack{(m,p)\in L \\ N < m+p \leqslant 2N-1}} \alpha(m+p) \sum_{j=0}^{a} \left(\frac{m}{N}\right)^{j} \left(\frac{m+p}{N} - 1\right)^{-(s+j)} + \sum_{(m,p)\in L} \alpha(m+p) \left(\frac{m}{N}\right)^{a+1} \sum_{q=0}^{\infty} \left(q + \frac{p}{N}\right)^{-1} \left(q + \frac{m+p}{N}\right)^{-(s+a)}\right].$$

Proceeding as in the analysis of H_1 with $s+1$ in place of s, we obtain a similar expression for H_2, which is meromorphic in the half-plane $\mathrm{Re}(s) > -(a+1)$,

$$H_2(\alpha, s) = \frac{N^{-(s+1)}}{2}\left[A(\alpha, s+1, a) - \sum_{\substack{(m,p)\in L \\ N < m+p \leqslant 2N-1}} \alpha(m+p) \sum_{j=0}^{a} \left(\frac{m}{N}\right)^{j} \left(\frac{m+p}{N} - 1\right)^{-(s+j+1)} + \sum_{(m,p)\in L} \alpha(m+p) \left(\frac{m}{N}\right)^{a+1} \sum_{q=0}^{\infty} \left(q + \frac{p}{N}\right)^{-1} \left(q + \frac{m+p}{N}\right)^{-(s+a+1)}\right].$$

Again, proceeding as in the analysis of H_1 with $s+2r$ in place of s, we obtain a similar expression for H_3, which is meromorphic in the half-plane

$\mathrm{Re}(s) > -(a+2)$,

$$H_3(\alpha, s) = N^{-(s+1)} \sum_{r=1}^{k} \binom{s+2r-2}{2r-1} \frac{B_{2r}}{2r}$$
$$\times \left[A(\alpha, s+2r, a) - \sum_{\substack{(m,p)\in L \\ N<m+p\leqslant 2N-1}} \alpha(m+p) \sum_{j=0}^{a} \left(\frac{m}{N}\right)^j \left(\frac{m+p}{N} - 1\right)^{-(s+j+2r)} \right.$$
$$\left. + \sum_{(m,p)\in L} \alpha(m+p) \left(\frac{m}{N}\right)^{a+1} \sum_{q=0}^{\infty} \left(q + \frac{p}{N}\right)^{-1} \left(q + \frac{m+p}{N}\right)^{-(s+a+2r)} \right].$$

Since $\{\alpha(n)\}$ is a bounded sequence and $P_{2k+1}(x)$ is bounded on $\mathbb{R}$, a straightforward estimate shows that H_4 is absolutely convergent for $\mathrm{Re}(s) > -2k$. Further, if we let $a = 2k$ in H_1, H_2 and H_3, then H_1, H_2, and H_3 are meromorphic in the half-plane $\mathrm{Re}(s) > -2k$. As these results hold for each integer $k \geqslant 1$, we conclude that $H(\alpha, s)$ has a meromorphic continuation to $\mathbb{C}$. □

3. Proofs of Theorem 1 and Corollary 2

In this section we prove Theorem 1 and Corollary 2.

Proof of Theorem 1. Let $l \geqslant 0$ be an integer, and let $k = l+1$ and $a = 2k$ in the expression $H(\alpha, s) = H_1 + H_2 + H_3 + H_4$ given in the second proof of Theorem 8. Then the vanishing of the binomial coefficients in H_3, respectively, H_4 for integers $r = l+1$, respectively, $l \geqslant 0$ yields

$$\lim_{s\to -2l} H(\alpha, s) = \lim_{s\to -2l} [H_1(\alpha, s) + H_2(\alpha, s) + H_3(\alpha, s)], \tag{3.1}$$

where r now runs from 1 to l in H_3. If $l = 0$, then H_3 is identically zero.

We want to show that $H_1 + H_2 + H_3$ is analytic at $s = -2l$, and then compute limit (3.1). First observe that the Hurwitz zeta functions in H_1, H_2 and H_3 corresponding to the integers $j = 2l+1$, $2l$ and $2l+1-2r$ for $r = 1, \ldots, l$, respectively, are given by

$$\zeta\left(s+2l+1, \frac{m+p}{N}\right) \quad \text{if } 0 < p+m \leqslant N$$

and

$$\zeta\left(s+2l+1, \frac{m+p}{N} - 1\right) \quad \text{if } N < p+m \leqslant 2N-1.$$

Therefore each of the functions H_1, H_2, and H_3 has only a simple pole at $s = -2l$. By taking the Laurent expansions of H_1, H_2 and H_3 at $s = -2l$, we determine the residues

$$\mathrm{Res}_{s=-2l} H_1(\alpha, s) = -\frac{N^{2l-1}}{2l+1} \sum_{(m,p)\in L} \alpha(m+p)\left(\frac{m}{N}\right)^{2l+1}, \tag{3.2}$$

$$\mathrm{Res}_{s=-2l} H_2(\alpha, s) = \frac{N^{2l-1}}{2} \sum_{(m,p)\in L} \alpha(m+p)\left(\frac{m}{N}\right)^{2l}, \tag{3.3}$$

and

$$\mathrm{Res}_{s=-2l} H_3(\alpha, s) = N^{2l-1} \sum_{r=1}^{l} \binom{2r-2l-2}{2r-1} \frac{B_{2r}}{2r} \sum_{(m,p)\in L} \alpha(m+p)\left(\frac{m}{N}\right)^{1+2l-2r}. \tag{3.4}$$

Using the periodicity of α, we obtain the following identity for each integer $b \geqslant 0$,

$$\begin{aligned} \sum_{(m,p)\in L} m^b \alpha(m+p) &= \left[\sum_{m=0}^{N-1} m^b\right] \sum_{p=1}^{N} \alpha(p) \\ &= \left[\sum_{r=0}^{b} (-1)^r \binom{b+1}{r} \frac{B_r (N-1)^{b+1-r}}{b+1}\right] \sum_{p=1}^{N} \alpha(p). \end{aligned} \tag{3.5}$$

Then Eq. (3.5) can be used to show that the sum of the residues in Eqs. (3.2)–(3.4) is zero. This proves that $H_1 + H_2 + H_3$ is analytic at $s = -2l$.

The leading term in the analytic part of the Laurent expansion of $\zeta(s + 2l + 1, x)$ at $s = -2l$ is $\gamma(x)$ (see [3, Theorem 12.21, p. 269]). Incorporating this fact into the Laurent expansions of H_1, H_2, and H_3 at $s = -2l$, we compute limit (3.1),

$$\lim_{s\to -2l} [H_1(\alpha, s) + H_2(\alpha, s) + H_3(\alpha, s)] = \sum_{d=1}^{6} g_d(\alpha, l, N),$$

where the terms $\{g_d\}$ are described below.

The term g_1 is given by

$$g_1(\alpha, l, N) = \sum_{\substack{(m,p)\in L \\ 0<m+p\leqslant N}} \alpha(m+p) \left[-\frac{N^{2l-1}}{2l+1} \sum_{\substack{j=0 \\ j\neq 2l+1}}^{2(l+1)} \left(\frac{m}{N}\right)^j \zeta\left(j-2l, \frac{m+p}{N}\right) \right.$$

$$+\frac{N^{2l-1}}{2}\sum_{\substack{j=0\\ j\neq 2l}}^{2(l+1)}\left(\frac{m}{N}\right)^{j}\zeta\left(j-2l+1,\frac{m+p}{N}\right)$$

$$\left.+N^{2l-1}\sum_{r=1}^{l}\binom{2r-2l-2}{2r-1}\frac{B_{2r}}{2r}\sum_{\substack{j=0\\ j\neq 1+2l-2r}}^{2(l+1)}\left(\frac{m}{N}\right)^{j}\zeta\left(j-2l+2r,\frac{m+p}{N}\right)\right].$$

The term g_2 is almost identical to g_1, the exceptions being that the outer sum is over $(m,p)\in L$ with $N<m+p<2N-1$, and the Hurwitz zeta functions are of the form $\zeta\left(s,\frac{m+p}{N}-1\right)$.

The term g_3 is given by

$$g_3(\alpha,l,N)=\sum_{\substack{(m,p)\in L\\ 0<m+p\leqslant N}}\alpha(m+p)\left[-\frac{N^{2l-1}}{2l+1}\left(\frac{m}{N}\right)^{2l+1}+\frac{N^{2l-1}}{2}\left(\frac{m}{N}\right)^{2l}\right.$$
$$\left.+N^{2l-1}\sum_{r=1}^{l}\binom{2r-2l-2}{2r-1}\frac{B_{2r}}{2r}\left(\frac{m}{N}\right)^{1+2l-2r}\right]\gamma\left(\frac{m+p}{N}\right).$$

Again, the term g_4 is almost identical to g_3, the exceptions being that the outer sum is over $(m,p)\in L$ with $N<m+p<2N-1$, and the expression inside the brackets multiplies $\gamma\left(\frac{m+p}{N}-1\right)$.

The term g_5 is given by

$$g_5(\alpha,l,N)=\sum_{\substack{(m,p)\in L\\ N<m+p\leqslant 2N-1}}\alpha(m+p)\sum_{j=0}^{2(l+1)}\left(\frac{m}{N}\right)^{j}$$
$$\times\left[\frac{N^{2l-1}}{2l+1}\left(\frac{m+p}{N}-1\right)^{-(j-2l)}-\frac{N^{2l-1}}{2}\left(\frac{m+p}{N}-1\right)^{-(j-2l+1)}\right.$$
$$\left.-N^{2l-1}\sum_{r=1}^{l}\binom{2r-2l-2}{2r-1}\frac{B_{2r}}{2r}\left(\frac{m+p}{N}-1\right)^{-(j+2r-2l)}\right].$$

Finally, the term g_6 is given by

$$g_6(\alpha,l,N)=\sum_{(m,p)\in L}\alpha(m+p)\left(\frac{m}{N}\right)^{2l+3}$$

$$\times\left[-\frac{N^{2l-1}}{2l+1}Z\left(2,\frac{p}{N},\frac{m+p}{N}\right)+\frac{N^{2l-1}}{2}Z\left(3,\frac{p}{N},\frac{m+p}{N}\right)\right.$$
$$\left.+N^{2l-1}\sum_{r=1}^{l}\binom{2r-2l-2}{2r-1}\frac{B_{2r}}{2r}Z\left(2r+2,\frac{p}{N},\frac{m+p}{N}\right)\right].\qquad\square$$

Proof of Corollary 2. First recall that $H(\alpha^0, s)$ has a pole at $s = 0$ (see Remark 9 above), thus when $\alpha = \alpha^0$, Eq. (1.1) is not valid at $s = 0$. However, Eq. (1.1) remains valid at $s = -2l$ for each integer $l \geqslant 1$. The condition $\alpha = \alpha^0$ forces $N = 1$, $m = 0$, $p = 1$, and

$$\{(m,p)\in L : N < m+p \leqslant 2N-1\} = \emptyset$$

in the right-hand side of Eq. (1.1), so that $g_d(\alpha^0, l, 1) = 0$ for $d = 2, \ldots, 6$, and in particular the only non-zero terms appear in $g_1(\alpha^0, l, 1)$ and correspond to $j = 0$. Using that $\zeta(-2l) = 0$ and $\zeta(1-2l) = -B_{2l}/2l$ for each integer $l \geqslant 1$ (see also Eqs. (5.1) and (5.3) below), and $\zeta(0) = -1/2$, we obtain

$$\begin{aligned} H(\alpha^0,-2l) &= -\frac{1}{2l+1}\zeta(-2l)+\frac{1}{2}\zeta(1-2l)+\binom{-2}{2l-1}\frac{B_{2l}}{2l}\zeta(0) \\ &= -\left[1+\binom{-2}{2l-1}\right]\frac{B_{2l}}{4l}.\qquad\square \end{aligned}$$

4. Meromorphic continuation of $F(\boldsymbol{\alpha}, s, x)$ and $G(\boldsymbol{\alpha}, s, x)$

In this section we prove that $F(\alpha, s, x)$ and $G(\alpha, s, x)$ have meromorphic continuations to $\mathbb{C}$. When $x = 1$ these results follow immediately from Eqs. (1.3) and (1.4), and Theorem 8.

Theorem 10. *The L-series $F(\alpha, s, x)$ has a meromorphic continuation to $\mathbb{C}$ with possible simple poles at $s = 0$ and $1 - 2r$ for each integer $r \geqslant 1$.*

Proof. For each integer $k \geqslant 1$, we have the generalized Stirling's formula (see [17, p. 530]),

$$\log(\Gamma(x)) = \left(x-\frac{1}{2}\right)\log(x) - x + \log(\sqrt{2\pi})$$
$$+\sum_{r=1}^{k}\frac{B_{2r}}{2r(2r-1)}x^{-(2r-1)} - (2k)!\int_0^\infty \frac{P_{2k+1}(t)}{(t+x)^{2k+1}}\,dt, \quad x>0. \quad (4.1)$$

Differentiating Eq. (4.1) with respect to x and simplifying yields the following form of the logarithmic derivative of the gamma function,

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log(x) - \frac{1}{2x} - \sum_{r=1}^{k} \frac{B_{2r}}{2rx^{2r}} + (2k+1)! \int_0^\infty \frac{P_{2k+1}(t)}{(t+x)^{2k+2}}\, dt. \tag{4.2}$$

Using the periodicity of α and Eq. (4.2), we obtain the following expression for $F(\alpha, s, x)$, valid for $\mathrm{Re}(s) > 0$,

$$F(\alpha, s, x) = -\frac{L(\alpha, s+1)}{2x} - \sum_{r=1}^{k} \frac{B_{2r}}{2rx^{2r}} L(\alpha, s+2r) + R(\alpha, k, s, x). \tag{4.3}$$

Here the remainder term is given by the series

$$R(\alpha, k, s, x) = \sum_{n=1}^{\infty} \alpha(n) \left[(2k+1)! \int_0^\infty \frac{P_{2k+1}(t)}{(t+nx)^{2k+2}}\, dt \right] n^{-s}.$$

Since $P_{2k+1}(t)$ is bounded on $\mathbb{R}$, a straightforward estimate shows that $R(\alpha, k, s, x)$ is absolutely convergent for $\mathrm{Re}(s) > -2k$. As this holds for each integer $k \geqslant 1$, we conclude from Eq. (4.3) that $F(\alpha, s, x)$ has a meromorphic continuation to $\mathbb{C}$ with possible simple poles at $s = 0$ and $1 - 2r$ for $r = 1, 2, \ldots$. $\square$

Proceeding as in Theorem 10, we can prove that $G(\alpha, s, x)$ has a meromorphic continuation to $\mathbb{C}$. The main difference is the appearance of $L'(\alpha, s)$ in the asymptotic formula for $G(\alpha, s, x)$, which contributes a possible double pole at $s = 1$. The details are left to the reader.

Theorem 11. *The L-series $G(\alpha, s, x)$ has a meromorphic continuation to $\mathbb{C}$ with a possible double pole at $s = 1$ and possible simple poles at $s = 0$ and $1 - 2r$ for each integer $r \geqslant 1$.*

5. Proofs of Theorems 4–6

In this section we prove Theorems 4–6. We begin by listing some facts concerning the Dirichlet L-function and generalized Bernoulli numbers. Facts 1 and 3 can be found in [16, Chapters 1 and 2], and Fact 2 can be found in [14, pp. 42–43].

1. Let χ be a primitive Dirichlet character of conductor N, χ^0 be the principal Dirichlet character of conductor 1, and $\chi \neq \chi^0$. Then the Dirichlet L-function $L(\chi, s)$ has an analytic continuation to an entire function on $\mathbb{C}$.

2. For each integer $l \geqslant 1$, define the lth Bernoulli polynomial,

$$B_l(x) = \sum_{j=0}^{l} \binom{l}{j} B_j x^{l-j} \in \mathbb{Q}[x]$$

and the lth generalized Bernoulli number,

$$B_{\chi,l} = \sum_{a=1}^{N} \chi(a) N^{l-1} B_l\left(\frac{a}{N}\right).$$

Then the following evaluation formula holds for each integer $l \geqslant 1$,

$$L(\chi, 1-l) = -\frac{B_{\chi,l}}{l}. \tag{5.1}$$

3. Define the symbol

$$\delta = \delta_\chi = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Then for each integer $l \geqslant 1$,

$$B_{\chi,l} \neq 0 \quad \text{if } l \equiv \delta \bmod 2, \tag{5.2}$$

$$B_{\chi,l} = 0 \quad \text{if } l \not\equiv \delta \bmod 2. \tag{5.3}$$

Next, recall that the Gauss sum of a Dirichlet character is defined by

$$\tau(\chi) = \sum_{a=1}^{N} \chi(a) e^{2\pi i a/N}, \quad i = \sqrt{-1}.$$

The following result can be extracted from Theorem 4.3, of Neukirch's article in the book [21, p. 214].

Theorem 12. (i) *Suppose $\chi \neq \chi^0$ with $\chi(-1) = 1$. For each integer $l \geqslant 0$,*

$$L'(\chi, -2l) = \frac{\tau(\chi)}{2} \frac{(2l)!}{(2\pi i)^{2l}} N^{2l} L(\bar{\chi}, 2l+1) \neq 0.$$

(ii) *For each integer $l \geqslant 1$,*

$$\zeta'(-2l) = \frac{1}{2} \frac{(2l)!}{(2\pi i)^{2l}} \zeta(2l+1) \neq 0.$$

Proof of Theorem 4. (A) Let $\alpha = \chi \neq \chi^0$ and $s = -2l$ in Eq. (1.3). Then depending on the sign of χ, both parts of (A) follow from applications of Theorem 12 (i), and Eqs. (5.1)–(5.3).

(B) Let $\alpha = \chi^0$ and $s = -2l$ in Eq. (1.3). Then (B) follows from applications of Theorem 12 (ii), Corollary 2, and Eqs. (5.1)–(5.3). $\square$

Using Eqs. (1.4) and (1.5), Theorems 5 and 6 can be proved in a manner analogous to Theorem 4. The details are left to the reader.

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